Kinetic roughening model with opposite Kardar-Parisi-Zhang nonlinearities

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We introduce a model that simulates a kinetic roughening process with two kinds of particle: one follows ballistic deposition (BD) kinetics and the other restricted solid-on-solid Kim-Kosterlitz (KK) kinetics. Both of these kinetics are in the universality class of the nonlinear Kardar-Parisi-Zhang equation, but the BD kinetics has a positive nonlinear constant while the KK kinetics has a negative one. In our model, called the BD-KK model, we assign the probabilities p and $(1-p)$ to the KK and BD kinetics, respectively. For a specific value of p, the system behaves as a quasilinear model and the up-down symmetry is restored. We show that nonlinearities of odd order are relevant in this low nonlinear limit.

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I. INTRODUCTION

The growth of interfaces by nonequilibrium kinetic roughening is a very interesting topic of far from equilibrium statistical mechanics [1–3]. In the two past decades, several models have been proposed, such as the ballistic deposition model [4], the Eden model [5], the solid-on-solid (SOS) model with surface relaxation [6,7], the SOS model with restriction [8], and the SOS model with diffusion [9]. In computer simulations, interfaces are described by a discrete set $\{h_i(t)\}$ that represents the height of site i at time t. Such an interface has L^d sites, where L is the linear size and d is the dimensionality of the substrate. The roughness of the interface is defined by

$$\omega^2(L,t)=\left\langle \frac{1}{L^d}\sum_{i=1}^{L^d}(h_i-\overline{h})^2\right\rangle, \tag{1}$$

where $\overline{h}$ is the mean height at time t and $\langle \cdots \rangle$ means the average over independent computational samples. In most of the kinetic roughening models, the roughness obeys the Family-Vicsek dynamical scaling [10]

$$\omega(L,t)\sim L^{\alpha}f\left(\frac{t}{L^z}\right), \tag{2}$$

where the function $f(x)$ must be L independent. The roughness behaves as $\omega\sim t^{\beta}$ for short times $(1\ll t\ll L^z)$ and as $\omega_\infty(L)\sim L^{\alpha}$ in the steady state. β and α are the growth and roughness exponents, respectively, and are related to the dynamical exponent z through the relation $z=\alpha/\beta$. For some systems, $\alpha=\beta=0$ and $z\neq 0$, which means that the roughness does not obey Eq. (2) and has a logarithmic behavior in space and time.

Kinetic growth models are also described by continuum Langevin-like equations in the coarse-grained limit. These equations have terms that represent the main interactions among the incoming particles. An example of a linear equation is the Edwards-Wilkinson (EW) equation [6],

$$\frac{\partial h(\mathbf{x},t)}{\partial t}=v_0+\eta(\mathbf{x},t)+\nu\nabla^2 h(\mathbf{x},t), \tag{3}$$

which describes the fluctuations of the SOS model with surface relaxation [7]. In Eq. (3), the first two terms on the right side are related to the deposition of particles. This deposition has a rate v_0 and a noise with zero mean and variance given by

$$\langle \eta(\mathbf{x},t)\eta(\mathbf{x}',t')\rangle = D\delta^d(\mathbf{x}-\mathbf{x}')\delta(t-t'). \tag{4}$$

The third term represents the surface relaxation process.

The exponents of Eq. (3), obtained by Fourier analysis [6,11], are $\beta(d)=(2-d)/4$, $\alpha(d)=(2-d)/2$, and $z(d)=2$. For $d=1$, these expressions give $\beta=1/4$ and $\alpha=1/2$. For $d=2$, the scaling exponents are $\beta=\alpha=0$.

There are also some nonlinear equations that describe nonlinear kinetic roughening models. The best known nonlinear equation is the Kardar-Parisi-Zhang (KPZ) equation [12]

$$\frac{\partial h(\mathbf{x},t)}{\partial t}=v_0+\eta(\mathbf{x},t)+\nu\nabla^2 h(\mathbf{x},t)+\frac{\lambda}{2}[\nabla h(\mathbf{x},t)]^2. \tag{5}$$

This equation is more complete than Eq. (4) because the nonlinear term may represent lateral growth or the appearance of a driven force. In $d=1$, the exponents of this equation [12] are $\beta=1/3$, $\alpha=1/2$, and $z=3/2$. In $d=2$, the analytical solution is not known. Examples of models in the universality class of the KPZ equation are the ballistic deposition (BD) model [4] and the SOS model with restriction [Kim-Kosterlitz (KK) model] [8]. In $d=1$, numerical simulations [13] indicate $\beta\approx 0.30$ and $\alpha\approx 0.47$ for the BD model and $\beta\approx 0.332$ and $\alpha\approx 0.489$ for the KK model. In $d=2$, these exponents are $\beta\approx 0.24$, $\alpha\approx 0.40$ (BD) and $\beta\approx 0.25$, $\alpha\approx 0.40$ (KK).

In this article we report on results of computer simulation of a growth model with two kinds of particle. Both of them obey KPZ kinetics, but they have opposite signs of the nonlinear constant λ. The effort to understand the nonlinearity in stochastic systems out of equilibrium is due to the great influence of the nonlinearities on the scaling analysis, as noted by Binder *et al.* [14]. The anisotropic KPZ equation, studied analytically in 1991 by Wolf [15], also contains two nonlinear terms with opposite signs that describe the distinguishable directions of the substrate in a vicinal surface. He found, by the renormalization group method, a logarithmic behavior of the roughness ($\alpha=\beta=0$) for opposite signs of the nonlinearity and power-law behavior when the nonlinearities have the same sign. These results were confirmed by Kim *et al.* [16] by simulations of a discrete model with two different kinetics applied one in each direction of the substrate. Another study along the same lines is the exact solution for a generalization of the Gates and Westcott model [17] to arbitrary inclinations obtained by Prähofer and Spohn [18]. They found exact temporal logarithmic growth of the height correlation, when the system has curvatures with opposite signs.

The motivation of our work is the paper published by Bernardes *et al.* [19]. They studied the deposition of particles with different radii on a cold substrate by Monte Carlo simulation. The authors found the growth exponent $\beta\approx 0.26$ for $d=1$. Thus they concluded that the universality class of this model is close to the EW class [6]. However, there are two nonlinear characteristics in the morphology of the model of Bernardes *et al.*: (i) porosity exists in the bulk; (ii) the growth velocity, that is, $v=\langle d\overline{h}/dt\rangle$, is greater than the deposition rate. These characteristics might indicate that the up-down symmetry ($h\rightarrow -h$) is broken [2]. The breaking of this symmetry leads to the appearance of the nonlinear term in the KPZ equation. Bernardes *et al.* also noted the need for a logarithmic correction in the behavior of the roughness [14,20], which indicates the presence of odd nonlinear terms in the growth equation. In the next section, we describe our model. In Sec. III, we show results and give some discussion. Finally, in Sec. IV, we show our conclusions.

II. MODEL

The aim of creating a model with opposite KPZ nonlinearities is to verify the possibility of generating a linear process (when the effective nonlinearity vanishes) with the morphology of a nonlinear growth process. Our model is a probabilistic combination of the ballistic deposition model and the SOS model with restriction (the KK model). The KK model occurs with *p* probability, and the BD model occurs with $(1-p)$ probability.

Ballistic deposition is a process where particles are dropped vertically onto a smooth substrate. The incoming particle is automatically joined to the growing cluster when its first contact with the growing interface occurs. In the in-lattice version, we select at random a site *i* of the lattice and its new height is evaluated by the algorithm [21]

$$h_i'=\max(h_i+1;h_{\{j\}}), \tag{6}$$

where $\{j\}$ are the first neighbors of the site *i*. BD kinetics does not generate solid-on-solid deposition because it generates a structure with porosity. Therefore, we define the growing profile as the maximum height of each column.

The KK model is a SOS random deposition with the difference of height constraint $h_i-h_{\{j\}}<m$, where *m* is the parameter that controls the roughness. If the height of the particle deposited on the site *i* does not satisfy the height constraint, this particle is not incorporated in the interface.

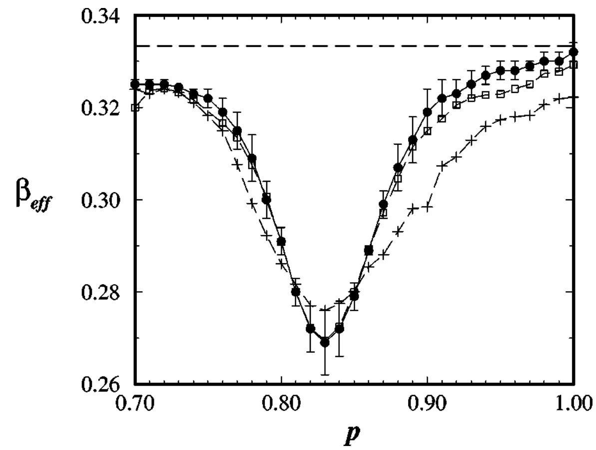

FIG. 1. The growth exponent β vs the parameter *p* of the BD-KK model for $L=50\,000$ (filled circles with error bars), $L=10\,000$ (open squares), and $L=2000$ (plus signs). The long-dashed line is the exact value of the β exponent for the KPZ equation.

Both models are in the same universality class as the KPZ equation. We have chosen these two models because (i) the BD model generates a bulk with porosity and has the nonlinear parameter $\lambda_{BD}>0$ because the growth velocity is bigger than the deposition rate v_0; (ii) the KK model has $\lambda_{KK}<0$ because the kinetics of restriction makes the growth velocity smaller than the rate of deposition.

In our simulations, a unit of time means that we have made *L* attempts at deposition. Moreover, all simulations were done with a one-dimensional substrate ($d=1$) and, in the KK model, the difference of height constraint $m=1$.

III. RESULTS AND DISCUSSION

Figure 1 shows a plot of the effective growth exponent β_{eff} vs the parameter *p*, for some system sizes *L*. The long-dashed line is the exact value of β obtained from the KPZ equation [12]. We have done 100 independent runs for each probability and each size. We note that a minimum occurs at $p^*\sim 0.83$ and this minimum is more pronounced as the system size grows. For $L=50\,000$ we have applied the consecutive slopes method [2] in the log-log plots of ω vs *t* in the time interval $20<t<10\,000$, giving an ensemble of β_{eff} exponents for each value of *p*. Thus, we have estimated the error bars around each value of β_{eff}. For $p^*=0.83$, the model is near to the EW class, because the growth exponent is $\beta_{eff}\approx 0.27$. However, the error bars in this region are larger, indicating the need for more careful analysis of the scaling. These results show that the effective KPZ term disappeared at p^*.

In order to better characterize the KPZ nonlinearity in our model, we did a finite size analysis on the growth velocity. In 1990, Krug and Meakin [22] showed that the finite size cor-

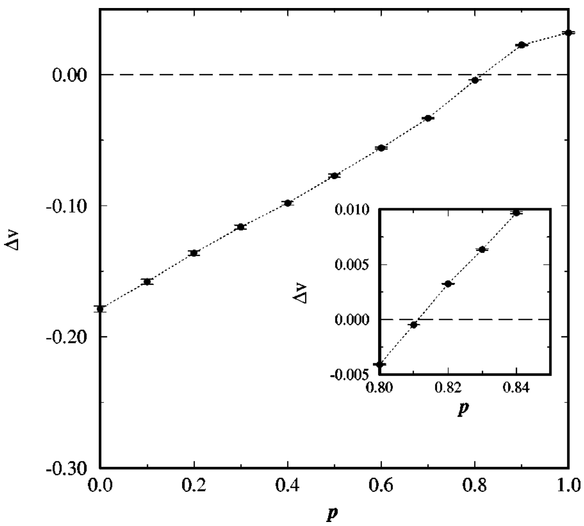

FIG. 2. Plot of the difference between the steady state growth velocities for $L=10$ and $L=1280$ vs the parameter p, which gives the amount of KPZ nonlinearity in the system. The inset shows the behavior close to the crossover.

rection $\Delta v(L,t)=v(L,t)-v_{\infty}$ for a model in the KPZ class behaves as

$$\Delta v(L)\sim-\lambda L^{-\alpha_{\parallel}} \quad \text{for} \quad t\gg L^{z}, \tag{7}$$

where the $\alpha_{\parallel}$ exponent depends on the roughness exponent. The Δv correction goes to zero when the KPZ term vanishes. So, with Eq. (7), we can obtain the sign of the KPZ nonlinearity and determine when the nonlinearity goes to zero as a function of the tuning parameter p. Figure 2 shows a plot of $\Delta v=v(L=10)-v(L=1280)$ vs p for the BD-KK model. The finite size correction vanishes for $p\approx0.81$ (see inset), very close to the value obtained at the minimum of the β vs p plot (see Fig. 1).

We also checked the need for multiplicative logarithmic corrections in the scaling for $p=p^{*}$. When the growth equation for a process has a sequence of odd nonlinear terms such as

$$\frac{\partial h(\mathbf{x},t)}{\partial t}=\eta(x,t)+\nu\frac{\partial^{2}h(\mathbf{x},t)}{\partial x^{2}}+\sum_{2n+1}\lambda_{2n+1}\left(\frac{\partial h(\mathbf{x},t)}{\partial x}\right)^{2n+1} \tag{8}$$

for $n=1,2,\ldots$, the roughness behaves as [14,20]

$$\omega(L,t)\sim t^{1/4}(\log t)^{1/8}, \tag{9}$$

for $t\ll L^{z}$. So, if logarithmic corrections are accepted for $p=p^{*}$, it means that the system is marginally in the EW class.

In order to show that Eq. (9) is really the best equation to describe the system near p^{*}, we rewrite this equation as

$$\omega(L,t)\sim t^{\delta}(\log t)^{\gamma}, \tag{10}$$

and we do small variations around the exact values $\delta=1/4$ and $\gamma=1/8$. We analyze the validity of Eq. (10) by evaluating the deviations from the horizontal curve $Y(\delta,\gamma,t)$ $=\omega(L,t)/t^{\delta}(\log t)^{\gamma}$ vs t, because this better emphasizes the deviations of the behavior from this equation. So we measure the relative error $\Delta Y/\langle Y\rangle$ of each curve $Y(\delta,\gamma,t)$ as a function of the variations in δ and γ. Figure 3(a) shows plots of $Y(t,\gamma)$ vs t with $\delta=1/4$ and Fig. 3(b) shows plots of $Y(t,\delta)$ vs t with $\gamma=1/8$. The insets show the relative error $\Delta Y/\langle Y\rangle$ vs the scaling exponent related to each case, γ or δ, respectively. The two bold curves show the behavior of $Y(\delta,\gamma,t)$ vs t when $\delta=1/4$ and $\gamma=1/8$. The relative error, for the two cases, reaches a minimum when the exponents $\gamma=1/8$ and $\delta=1/4$, indicating that Eq. (9) is a good description of the temporal behavior of the roughness. This indicates that odd nonlinear terms are relevant.

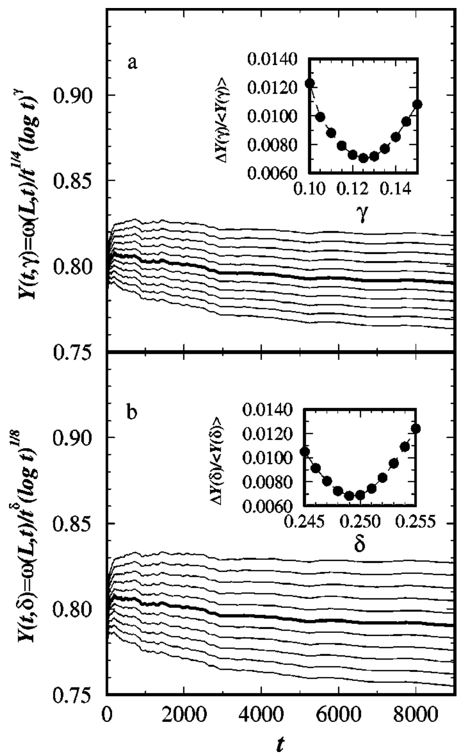

FIG. 3. Plots of the function $Y(t,\gamma,\delta)$ vs the time t: (a) $\delta=1/4$ and small variations in γ are performed, (b) $\gamma=1/8$ and small variations in δ are performed. The bold curves show the behavior of the function $Y(t,1/8,1/4)$. The insets show the relative error $\Delta Y/\langle Y\rangle$ vs the scaling exponent related to each case.

The nonlinear Eq. (8) preserves the up-down symmetry, but this is not obvious in the BD-KK model at $p=p^{*}$, nor in the growth model of Bernardes *et al.* Equation (1) can be generalized for any moment q of the height distribution, in $d=1$, as

$$\omega^{q}(L,t)=\left\langle\frac{1}{L}\sum_{i=1}^{L}(h_{i}-\overline{h})^{q}\right\rangle, \tag{11}$$

and we concentrate on the behavior of odd moments, especially the third moment $(q=3)$, which is related to up-down symmetry. The skewness, defined by $\mathcal{S}=\omega^{3}/(\omega^{2})^{3/2}$, can show us if the system has this symmetry. If the skewness is null, the system has up-down symmetry because all odd mo-

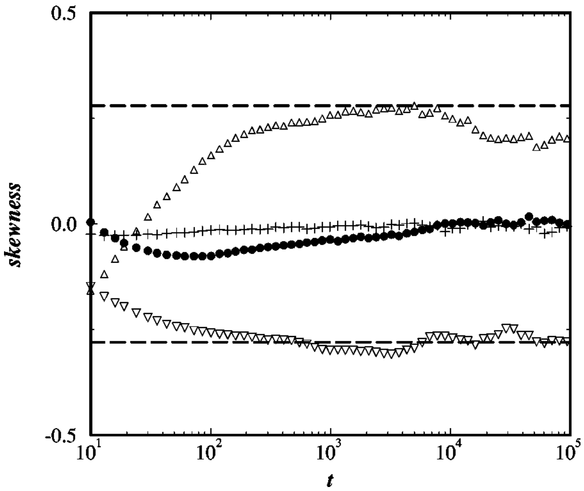

FIG. 4. Plots of the skewness $\mathcal{S}$ as a function of the time t for $p=0.0$ (up triangles), $p=1.0$ (down triangles), and $p=p^*$ (filled circles). The skewness for the EW linear model is represented by the plus symbols. The long-dashed straight lines indicate the ± 0.28 estimated values. All simulations were done with $L=50\,000$.

ments of the height distribution vanish. On the other hand, for systems without up-down symmetry and in the KPZ class, the steady skewness has a universal value $|\mathcal{S}|\approx 0.28$ [23]. Figure 4 shows the skewness $\mathcal{S}$ as a function of the time t for the model with positive nonlinearity ($p=0.0$, up triangles), with negative nonlinearity ($p=1.0$, down triangles), and at the low nonlinear point $p=p^*$ (filled circles). As illustration, we also show the curve for the model with surface relaxation [7], which is a linear model and has up-down symmetry (pluses). We note that the skewness for $p=0$ tends to $\mathcal{S}=0.28$ and, for $p=1.0$, to $\mathcal{S}=-0.28$. For $p=p^*$, in the asymptotic limit, the skewness goes to zero, suggesting the presence of up-down symmetry.

IV. CONCLUSIONS

We have studied the scaling properties of a model with opposite signs of the KPZ nonlinearity through numerical simulations of a model with two kinds of particle. Kim-Kosterlitz kinetics occurs with the probability p and the ballistic deposition model with the probability $(1-p)$. For a specific value of the tuning parameter $p=p^*$, we show that the KPZ nonlinearity goes to zero and the up-down symmetry is restored, but there are morphological characteristics of nonlinearity, such as lateral growth, due to the presence of ballistic deposition. We also show that logarithmic corrections are well adjusted for the temporal behavior of the system at $p=p^*$. We have done a careful study of the exponents of Eq. (10) and obtained $\delta=1/4$ and $\gamma=1/8$ as the best values for these parameters, showing the need for the logarithmic corrections and consequently the relevance of odd nonlinear terms in the coarse-grained limit.

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